7.3 - Operational Properties I

Shifting

Theorem: First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$.

Pf: By def 7.1.1, $\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{at}e^{\underline{-st}}f(t)\,dt$

$= \int_0^\infty e^{-\underline{(s-a)}t}f(t)\,dt = F(s-a)$

$-(s-a)$ is playing the role of $-s$

Notation: $\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f(t)\}_{s \to s-a}$

Inverse rule: $\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

Example: Find either $F(s)$ or $f(t)$, as indicated.

$\mathscr{L}\{t^{10}\underline{e^{-7t}}\}$

$$\boxed{\frac{10!}{(s+7)^{11}}}$$

$\mathscr{L}\{1\cdot e^{-at}\}$

$\downarrow$

$$\frac{1}{s+a}$$

$\mathscr{L}\{\underline{e^{-2t}}\,\underline{\cos 4t}\}$

$$\boxed{\frac{s+2}{(s+2)^2+16}}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+2s+\underset{1+4}{\underline{5}}}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2+2s+1+4}\right\}$$

$$= \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1\cdot 2}{\underline{(s+1)}^2+4}\right\} = \boxed{\frac{1}{2}e^{-t}\sin 2t}$$

shift $\to -1$ $\qquad$ $k=2$

$$\mathscr{L}^{-1}\left\{\frac{(s+1)^2}{\underset{\text{Shift } a=-2}{(s+2)}^{\overset{n=3}{4}}}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2+2s+1}{(s+2)^4}\right\}$$

$s^2+2s = s(s+2)$

$$= \mathcal{L}^{-1}\left\{\frac{s^2}{(s+2)^4} + \frac{2s}{(s+2)^4} + \frac{1}{(s+2)^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^3} + \frac{1}{(s+2)^4}\right\}$$

PFD

constant

$$\frac{s}{(s+2)^3} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$$

linear base

$$s = \cancel{A(s+2)^2} + B(s+2) + C$$

$s^2: \quad 0 = A$ $\qquad s: \quad 1 = B$ $\qquad$ Const: $0 = 2B + C$

$C = -2$

$$\mathcal{L}\left\{\frac{(s+1)^2}{(s+2)^4}\right\} = \left\{\frac{1}{(s+2)^{\underset{n=1}{2}}} - \frac{\overset{n!}{2}}{(s+2)^{\underset{n=2}{3}}} + \frac{1}{6}\,\frac{1\cdot 6}{(s+2)^{\underset{n=3}{4}}}\right\}$$

$$= \boxed{te^{-2t} - t^2e^{-2t} + \frac{1}{6}t^3e^{-2t}}$$

Example: Use the Laplace transform to solve the initial-value problem.

$y' - y = 1 + \boxed{t}e^t, \qquad y(0) = 0$

$$sY(s) - Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2}$$

$$(s-1)Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2}$$

$$Y(s) = \frac{1}{s(s-1)} + \frac{1}{(s-1)^3}$$

PFD

n=2 with a shift

$$Y(s) = -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{2}\frac{1\cdot 2}{(s-1)^3}$$

$$\boxed{y(t) = -1 + e^t + \frac{1}{2}t^2e^t}$$

cos

sine

$$\frac{s-a + b}{\cancel{s}^2 + k^2} \qquad (s-a)^2$$

$2y'' + 20y' + 51y = 0, \qquad y(0) = 2, \qquad y'(0) = 0$

$$2s^2Y(s) - 4s + 20sY(s) - 40 + 51Y(s) = 0$$

$$(2s^2 + 20s + 51)Y(s) = 4s + 40$$

$$Y(s) = \frac{4s+40}{2s^2+20s+51} = \frac{2s+20}{s^2+10s+\frac{51}{2}}$$

$25 + \frac{1}{2}$

$$Y(s) = \frac{2s+20}{(s+5)^2 + \frac{1}{2}}$$

shift

sine and/or cosine $\left(k = \frac{1}{\sqrt{2}}\right)$

$2(s+10)$

$$Y(s) = 2\left(\frac{s+5}{(s+5)^2+\frac{1}{2}} + \sqrt{2}\,\frac{5\cdot\frac{1}{\sqrt{2}}}{(s+5)^2+\frac{1}{2}}\right)$$

$$\boxed{y(t) = 2e^{-5t}\cos\frac{1}{\sqrt{2}}t + 10\sqrt{2}\,e^{-5t}\sin\frac{1}{\sqrt{2}}t}$$

$$\frac{s+3-3}{(s+3)^2+4} \rightarrow \frac{s+3}{(s+3)^2+4} - \frac{3}{(s+3)^2+4}$$

Example: Solve the boundary-value problem.

$$y'' + 8y' + 20y = 0, \qquad y(0) = 0, \quad y'(\pi) = 0$$

Let $c = y'(0)$

$$s^2Y(s) - s\overset{0}{\cancel{y(0)}} - y'(0) + 8sY(s) - 8\overset{0}{\cancel{y(0)}} + 20Y(s) = 0$$

$$(s^2+8s+20)Y(s) = c$$

$$(s+4)^2+4$$

$$Y(s) = \frac{1}{2}\,\frac{c\cdot 2}{(s+4)^2+4}$$

$$y(t) = \frac{1}{2}c\,e^{-4t}\sin 2t$$

$$y'(t) = \frac{c}{2}\left(-4e^{-4t}\overset{0}{\cancel{\sin 2t}} + 2e^{-4t}\cos 2t\right)$$

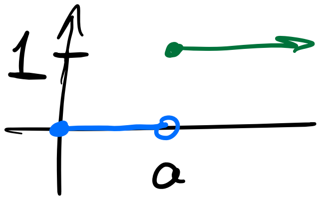
$$y'(\pi) = 0 \Rightarrow y'(\pi) = \frac{c}{2}\left(2e^{-4\pi}\right) \Rightarrow c = 0$$

$$\boxed{y = 0}$$

The First Translation Theorem showed how to translate on the s-axis. A translation on the t-axis is essentially an adjustment for moving the lower limit in the definition of a Laplace transform. This is accomplished via the **Heaviside function** or **unit step function.**

Definition: The **unit step function** is

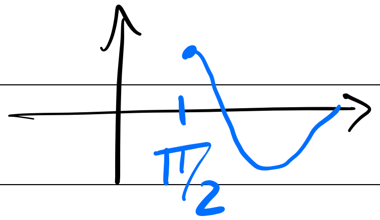
$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \quad \text{off} \\ 1, & t \ge a \quad \text{on} \end{cases}$$



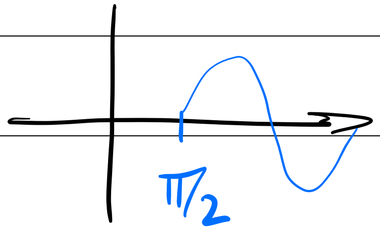
To make it first on and then off:

$$1-\mathscr{U}(t-a) = \begin{cases} 1-0, & 0 \le t < a \\ 1-1 & t \ge a \end{cases} = \begin{cases} 1, & 0 \le t < a \\ 0, & t \ge a \end{cases}$$

Consider $\sin t\,\mathscr{U}(t-\pi/2) = \begin{cases} 0, & 0 \le t < \pi/2 \\ \sin t, & t \ge \pi/2 \end{cases}$



$\sin(t-\pi/2)\,\mathscr{U}(t-\pi/2)$ yields



Theorem: Second Translation Theorem

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then $\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$.

pf: $\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,\mathcal{U}(t-a)\,dt$

$= \int_0^a e^{-st} f(t-a)\,0\,dt + \int_a^\infty e^{-st} f(t-a)\,1\,dt$

v-sub: $v = t-a \Rightarrow dv = dt$

$t = v+a$

$= \int_0^\infty e^{-s(v+a)} f(v)\,dv = e^{-as}\int_0^\infty e^{-sv} f(v)\,dv$

$= e^{-as}F(s)$ ✓

this is the shift

If we let $f(t) = 1$, then we have $\mathcal{L}\{\mathcal{U}(t-a)\} = \dfrac{e^{-as}}{s}$.

Example: Find either $F(s)$ or $f(t)$, as indicated.

$\mathcal{L}\{(3t+1)\mathcal{U}(t-1)\}$

Alternative form of the second translation theorem:

$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

original: $\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}F(s)$

Let $g(t) = f(t-a) \Rightarrow f(t) = g(t+a)$

Then $\mathcal{L}\{g(\underline{t})\,\mathcal{U}(\underline{t-a})\} = e^{-as}\mathcal{L}\{g(t+a)\}$

these don't match

compensation

$\mathcal{L}\{(3t+1)\,\mathcal{U}(t-1)\} = e^{-s}\mathcal{L}\{3(t+1)+1\}$

$= e^{-s}\mathcal{L}\{3t+4\}$

$$F(s) = e^{-s}\left(\frac{3}{s^2} + \frac{4}{s}\right)$$

Inverse form of the Second Translation Theorem:

If $f(t) = \mathscr{L}^{-1}\{F(s)\}$, then $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$.

Example: Find either $F(s)$ or $f(t)$, as indicated.

$$\mathscr{L}^{-1}\left\{\frac{(1+e^{-2s})^2}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1+2e^{-2s}+e^{-4s}}{s+2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+2} + \frac{2e^{-2s}}{s+2} + \frac{e^{-4s}}{s+2}\right\}$$

$$\boxed{f(t) = e^{-2t} + 2e^{-2(t\underline{-2})}\mathcal{U}(t\underline{-2}) + e^{-2(t-4)}\mathcal{U}(t-4)}$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\}$$

Example: Use the Laplace transform to solve the given initial-value problem.

$y' + y = f(t), \qquad y(0) = 0$, where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ -1 & t \geq 1 \end{cases}$$